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Classical solutions of the degenerate Garnier system and their coalescence structures

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Abstract
We study the degenerate Garnier system which generalizes the fifth Painlevé equation $P_{\rm V}$. We present two classes of particular solutions, classical transcendental and algebraic ones. Their coalescence structure is also investigated.

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1. Introduction

The Painlevé equations P_J ($J = \mathrm{I}, \ldots, \mathrm{VI}$) are derived from the theory of monodromy preserving deformations of linear differential equations of the form

$$(L_J) \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p_1(x,t)\frac{\mathrm{d}y}{\mathrm{d}x} + p_2(x,t)y = 0,$$

with singularities corresponding to a partition of four as follows (see e.g. [2]):

$L_{\rm VI}$	(1, 1, 1, 1)
$L_{\rm V}$	(1, 1, 2)
$L_{\rm IV}$	(1, 3)
$L_{\rm III}$	(2, 2)
$L_{\rm II}$	(4)

In this table, a partition $(r_1, \ldots, r_k)$ indicates that L_J has k singularities of Poincaré ranks $r_1 - 1, \ldots, r_k - 1$, respectively. Thus we regard each of P_J ($J = \mathrm{II}, \ldots, \mathrm{VI}$) as an equation corresponding to a partition of four. We note that the length k of the partition equals the number of constant parameters contained in P_J. The first Painlevé equation $P_{\rm I}$ has no constant parameter and does not correspond to any partition.

The Garnier system (in N variables) generalizes the sixth Painlevé equation P_{VI} and governs the monodromy preserving deformation of linear differential equation with $N+3$ regular singularities [2]. We also regard the Garnier system as corresponds to the partition $(1,\ldots,1)$ of $N+3$.

Each of the Painlevé equations P_J ($J = \mathrm{I},\ldots,\mathrm{V}$) can be reduced from the sixth one through a certain limiting procedure, in parallel with the confluence of singularities of the linear differential equation L_J [14]. Similarly, the degenerations of the Garnier system are considered [5–7, 10, 16]. Each of them is associated with a partition. We denote by $G(r_1,\ldots,r_k;N)$ the degenerate Garnier system in N variables corresponding to a partition $(r_1,\ldots,r_k)$ of $N+3$.

It is well known that each of P_J ($J = \mathrm{II},\ldots,\mathrm{VI}$) admits two classes of classical solutions, hypergeometric and algebraic (or rational) ones. The coalescence structure of these solutions is investigated in detail [11, 12], as well as the degeneration scheme of the Painlevé equations. Also, the Garnier system $G(1,\ldots,1;N)$ has such classes of classical solutions [8, 18–20]. The aim of this paper is to study particular solutions of the degenerate Garnier system $G(1,\ldots,1,2;N)$ which generalizes the fifth Painlevé equation P_{V} and their coalescence structure by means of τ-functions.

We have in [17] a family of τ-functions for $G(1,\ldots,1;N)$ arranged on a lattice. This family is determined by a certain completely integrable Pfaffian system. In section 2, we investigate the degeneration of the Pfaffian system together with the degenerate limiting procedure from $G(1,\ldots,1;N)$ to $G(1,\ldots,1,2;N)$; hence we obtain a family of τ-functions on a lattice for $G(1,\ldots,1,2;N)$. We have in particular (see theorems 3.2, 3.3 and 4.2) the following.

Theorem 1.1. *The system $G(1,\ldots,1,2;N)$ admits three types of solutions:*

(i) classical transcendental ones expressed by the hypergeometric series Φ_D;
(ii) rational ones in terms of the Schur polynomials;
(iii) algebraic ones in terms of the universal characters.

2. Degenerate Garnier system

In this section, we formulate the degenerate Garnier system $G(1,\ldots,1,2;N)$, then introduce a family of τ-functions for the system.

2.1. Hamiltonian system and Schlesinger system

Let $\{,\}$ be the Poisson bracket defined by

$$\{f,g\} = \sum_{j=1}^{N}\left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial g}{\partial p_j}\frac{\partial f}{\partial q_j}\right). \tag{2.1}$$

Consider the following completely integrable Hamiltonian system:

$$\mathrm{d}q_j = \sum_{i=1}^{N}\{K_i,q_j\}\,\mathrm{d}s_i, \qquad \mathrm{d}p_j = \sum_{i=1}^{N}\{K_i,p_j\}\,\mathrm{d}s_i \quad (j=1,\ldots,N), \tag{2.2}$$

with polynomial Hamiltonians K_i ($i = 1, \ldots, N$):

$$\begin{aligned}
s_1^2 K_1 = {}& q_1 \left(\rho + \sum_{j=1}^{N} q_j p_j\right) \left(\rho + \theta_{N+3} + 1 + \sum_{j=1}^{N} q_j p_j\right) \\
& + \sum_{j=2}^{N} s_1 p_1 q_j - \sum_{j=2}^{N} s_j q_1 (q_j p_j - \theta_j) p_j - \sum_{j=2}^{N} (s_j - 1) q_j p_j \\
& - s_1 q_1 p_1 (q_1 p_1 - \theta_{N+2}) + (q_1 - s_1) p_1, \\
s_i(s_i - 1) K_i = {}& q_i \left(\rho + \sum_{j=1}^{N} q_j p_j\right) \left(\rho + \theta_{N+3} + 1 + \sum_{j=1}^{N} q_j p_j\right) \\
& - \sum_{j=2, j\neq i}^{N} R_{ij} q_i p_i (q_j p_j - \theta_j) - \sum_{j=2, j\neq i}^{N} R_{ji} q_i (q_j p_j - \theta_j) p_j \\
& - \sum_{j=2, j\neq i}^{N} S_{ij} p_i (q_i p_i - \theta_i) q_j - \sum_{j=2, j\neq i}^{N} R_{ij} (q_i p_i - \theta_i) q_j p_j \\
& + \{s_i p_i - (s_i + 1) q_i p_i\} (q_i p_i - \theta_i) + (\theta_{N+2} s_i + \theta_{N+1} - 1) q_i p_i \\
& + \frac{s_i(s_i - 1)}{s_1} \{q_i p_i + p_i (q_i p_i - \theta_i) q_1\} - (s_i - 1) q_i p_1 \\
& - s_i (2 q_i p_i - \theta_i) q_1 p_1 \qquad (i = 2, \ldots, N),
\end{aligned} \tag{2.3}$$

where

$$\sum_{j=2}^{N+3} \theta_j + 2\rho = 0, \tag{2.4}$$

and

$$R_{ij} = \frac{s_i(s_j - 1)}{s_j - s_i}, \qquad S_{ij} = \frac{s_i(s_i - 1)}{s_i - s_j}. \tag{2.5}$$

We call (2.2) the *degenerate Garnier system* and denote it by $G(1, \ldots, 1, 2; N)$. This system is regarded as a generalization of the fifth Painlevé equation P_V [15]. For $N = 1$, this is exactly the Hamiltonian system of P_V. We note that $G(1, \ldots, 1, 2; N)$ is equivalent to the system given by Kimura [7] via a certain canonical transformation.

Let A_j ($j = 1, \ldots, N + 2$) be matrices of the dependent variables defined by

$$A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}. \tag{2.6}$$

Consider the following system of differential equations:

$$\begin{aligned}
\mathrm{d}A_1 &= \sum_{i=2}^{N+1} [A_i, A_1]\, \mathrm{d}\log t_i + (A_1 + [A_{N+2}, A_1])\, \mathrm{d}\log t_1, \\
\mathrm{d}A_j &= \sum_{i=2, i\neq j}^{N+2} [A_i, A_j]\, \mathrm{d}\log(t_j - t_i) + \frac{[A_1, A_j]}{t_j}\, \mathrm{d}\log \frac{t_j}{t_1} \qquad (j = 2, \ldots, N + 1), \\
\mathrm{d}A_{N+2} &= \sum_{i=2}^{N+1} \left(\frac{[A_i, A_1]}{t_i}\, \mathrm{d}\log \frac{t_i}{t_1} + [A_i, A_{N+2}]\, \mathrm{d}\log t_i \right),
\end{aligned} \tag{2.7}$$

where $t_{N+1} = 1$ and $t_{N+2} = 0$. Here we assume

(i) $\operatorname{tr} A_1 = t_1, \operatorname{tr} A_j = \theta_j \notin \mathbb{Z} \quad (j = 2, \ldots, N+2)$;
(ii) $\det A_j = 0\,(j = 1, \ldots, N+1), \operatorname{tr} A_1 A_{N+2} = t_1\theta_{N+2}$;
(iii) The matrices A_j satisfy

$$A_\infty := -\sum_{j=2}^{N+2} A_j = \begin{pmatrix} \rho & 0 \\ 0 & \rho + \theta_{N+3} \end{pmatrix}, \quad \theta_{N+3} \notin \mathbb{Z}. \tag{2.8}$$

We call (2.7) the *degenerate Schlesinger system* denoted by $S(1, \ldots, 1, 2; N)$.

The system $S(1, \ldots, 1, 2; N)$ is in fact equivalent to $G(1, \ldots, 1, 2; N)$ via

$$\begin{aligned} s_1 &= -\frac{1}{t_1}, & s_i &= \frac{t_i - 1}{t_i}, \\ q_1 &= -\frac{b_1}{t_1 b_\infty}, & q_i &= (t_i - 1)\frac{b_i}{b_\infty}, \\ q_1 p_1 &= a_1 + a_{N+2} - b_1 \frac{a_{N+1}}{b_{N+1}} - b_{N+2}\frac{a_1}{b_1}, \\ q_i p_i &= a_i - t_i b_i \frac{a_{N+1}}{b_{N+1}} + (t_i - 1) b_i \frac{a_1}{b_1} \quad (i = 2, \ldots, N), \end{aligned} \tag{2.9}$$

where $b_\infty = b_1 + \sum_{j=2}^{N+2} t_j b_j$.

Recall that both of $G(1, \ldots, 1, 2; N)$ and $S(1, \ldots, 1, 2; N)$ govern the holonomic deformation of the system of linear differential equations

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}x} = A(x, t)\vec{y}, \qquad A(x, t) = \frac{A_1(t)}{x^2} + \sum_{j=2}^{N+2} \frac{A_j(t)}{x - t_j}, \tag{2.10}$$

concerning the parameter $t = (t_1, \ldots, t_N)$, see [3].

2.2. *A family of τ-functions*

Proposition 2.1 ([4]). *For each solution of $S(1, \ldots, 1, 2; N)$, the 1-form*

$$\omega_0 = \sum_{i=1}^{N} H_i \, \mathrm{d}t_i, \tag{2.11}$$

is closed. Here we let

$$\begin{aligned} H_1 &= -\frac{1}{t_1} \det A_{N+2} - \sum_{j=2}^{N+1} \frac{\operatorname{tr} A_1 A_j - t_1\theta_j}{t_1 t_j}, \\ H_i &= \frac{\operatorname{tr} A_i A_1 - t_1\theta_i}{t_i^2} + \sum_{j=2, j\neq i}^{N+2} \frac{\operatorname{tr} A_i A_j - \theta_i\theta_j}{t_i - t_j} \quad (i = 2, \ldots, N). \end{aligned} \tag{2.12}$$

Proposition 2.1 allows us to define the τ-function $\tau_0 = \tau_0(t)$ by

$$\mathrm{d} \log \tau_0 = \omega_0, \tag{2.13}$$

up to multiplicative constants.

Let L_2 be a subset of $\mathbb{Z}^{N+2}$ defined as

$$L_2 = \{\nu = (\nu_2, \ldots, \nu_{N+3}) \in \mathbb{Z}^{N+2} | |\nu| = \nu_2 + \cdots + \nu_{N+3} \in 2\mathbb{Z}\}. \tag{2.14}$$

Then $S(1,\ldots,1,2;N)$ is invariant under the action of the Schlesinger transformations T_ν $(\nu \in L_2)$ which act on the parameters as follows (see [4]):

$$T_\nu(\theta_j) = \theta_j + \nu_j \quad (j = 2,\ldots,N+3). \tag{2.15}$$

We give explicitly the action of T_ν on the dependent variables in the appendix A.

Let us define a family of τ-functions by

$$d\log\tau_\nu = T_\nu(\omega_0) \quad (\nu \in L_2). \tag{2.16}$$

Remark 2.2. A family of τ-functions for $S(1,\ldots,1,2;N)$ can be identified with that for $G(1,\ldots,1,2;N)$ by

$$\sum_{i=1}^{N} K_i\,\mathrm{d}s_i = T_{(0,\ldots,0,1,0,-1)}(\omega_0). \tag{2.17}$$

Conversely, we can express a solution of $S(1,\ldots,1,2;N)$ in terms of τ-functions as follows. By

$$T_{(0,\ldots,0,2)}(H_i) = H_i + D_i\log b_\infty \quad (i = 1,\ldots,N), \tag{2.18}$$

where $D_i = \partial/\partial t_i$, we obtain

Proposition 2.3. *A solution of $S(1,\ldots,1,2;N)$ is expressed by means of τ-functions as follows:*

$$\begin{aligned}
a_1 &= \frac{t_1}{\theta_{N+3}}(D_1D_{N+3}\log\tau_0 - \rho), \qquad b_1 = t_1D_1\frac{\tau_{(0,\ldots,0,2)}}{\tau_0},\\
a_i &= \frac{1}{\theta_{N+3}}(D_iD_{N+3}\log\tau_0 - \theta_i\rho), \qquad b_i = D_i\frac{\tau_{(0,\ldots,0,2)}}{\tau_0} \quad (i = 2,\ldots,N),\\
a_{N+1} &= \frac{1}{\theta_{N+3}}\{(D_{N+1}+1)D_{N+3}\log\tau_0 - \rho(\rho+\theta_{N+1}+\theta_{N+3})\},\\
b_{N+1} &= (D_{N+1}+\theta_{N+3}+1)\frac{\tau_{(0,\ldots,0,2)}}{\tau_0},\\
a_{N+2} &= \frac{1}{\theta_{N+3}}\{(D_{N+2}-1)D_{N+3}\log\tau_0 - \rho(\rho+\theta_{N+2}+\theta_{N+3})\},\\
b_{N+2} &= (D_{N+2}-\theta_{N+3}-1)\frac{\tau_{(0,\ldots,0,2)}}{\tau_0},
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
D_{N+1} &= -\sum_{i=1}^{N} t_iD_i, \qquad D_{N+2} = t_1D_1 + \sum_{j=2}^{N}(t_j-1)D_j,\\
D_{N+3} &= -t_1D_1 + \sum_{i=2}^{N} t_i(t_i-1)D_i.
\end{aligned} \tag{2.20}$$

2.3. Coalescence structures

As is known in [2], the Garnier system $G(1,\ldots,1;N)$ is equivalent to the Schlesinger system, denoted by $S(1,\ldots,1;N)$

$$\mathrm{d}A_j = \sum_{i=1,i\neq j}^{N+2}[A_i, A_j]\,\mathrm{d}\log(t_j - t_i), \quad (j = 1,\ldots,N+2), \tag{2.21}$$

with the following conditions:

(i) $\det A_j = 0, \operatorname{tr} A_j = \theta_j \notin \mathbb{Z} \quad (j = 1, \ldots, N+2)$;
(ii) The matrices A_j satisfy

$$A_\infty := -\sum_{j=1}^{N+2} A_j = \begin{pmatrix} \rho & 0 \\ 0 & \rho + \theta_{N+3} \end{pmatrix}, \qquad \theta_{N+3} \notin \mathbb{Z}. \tag{2.22}$$

Let L_1 be a subset of $\mathbb{Z}^{N+3}$ defined as

$$L_1 = \{\mu = (\mu_1, \ldots, \mu_{N+3}) \in \mathbb{Z}^{N+3} | |\mu| = \mu_1 + \cdots + \mu_{N+3} \in 2\mathbb{Z}\}. \tag{2.23}$$

Then a family of τ-functions for $S(1, \ldots, 1; N)$ is defined by

$$\mathrm{d} \log \tau_\mu = \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N+2} \frac{1}{t_i - t_j} T_\mu (\operatorname{tr} A_i A_j - \theta_i \theta_j) \, \mathrm{d}t_i \quad (\mu \in L_1). \tag{2.24}$$

Here we let T_μ be the Schlesinger transformations given in [17].

The system $S(1, \ldots, 1, 2; N)$ is obtained from $S(1, \ldots, 1; N)$ by the replacement

$$\begin{aligned} &\theta_1 \to 1/\varepsilon, \qquad \theta_{N+2} \to \theta_{N+2} - 1/\varepsilon, \qquad t_1 \to \varepsilon t_1, \\ &A_1 \to \frac{A_1}{\varepsilon t_1}, \qquad A_{N+2} \to A_{N+2} - \frac{A_1}{\varepsilon t_1}, \end{aligned} \tag{2.25}$$

and taking a limit $\varepsilon \to 0$. Then (2.24) is also transformed into (2.16) via

$$\tau_\mu \to \tau_\nu \quad (\mu \in L_1), \tag{2.26}$$

where

$$\nu = (\mu_2, \ldots, \mu_{N+1}, \mu_1 + \mu_{N+2}, \mu_{N+3}) \in L_2. \tag{2.27}$$

3. Classical transcendental solutions

In this section, a family of classical transcendental solutions is presented. This is reduced to a family of rational solutions expressed in terms of the Schur polynomials.

We recall the definition of the Lauricella hypergeometric series F_D. For each $m = (m_1, \ldots, m_N)$, we let

$$t^m = t_1^{m_1} \cdots t_N^{m_N}, \qquad |m| = m_1 + \cdots + m_N. \tag{3.1}$$

The series F_D is defined by

$$F_D(\alpha, \beta_1, \cdots, \beta_N, \gamma; t) = \sum_{m \in (\mathbb{Z}_{\geqslant 0})^N} \frac{(\alpha)_{|m|} (\beta_1)_{m_1} \cdots (\beta_N)_{m_N}}{(\gamma)_{|m|} (1)_{m_1} \ldots (1)_{m_N}} t^m, \tag{3.2}$$

where

$$(\alpha)_k = \alpha(\alpha+1) \cdots (\alpha + k - 1). \tag{3.3}$$

Via (2.25) and taking a limit $\varepsilon \to 0$, F_D is transformed into

$$\Phi_D(\alpha, \beta_2, \ldots, \beta_N, \gamma; t) = \sum_{m \in (\mathbb{Z}_{\geqslant 0})^N} \frac{(\alpha)_{|m|} (\beta_2)_{m_2} \cdots (\beta_N)_{m_N}}{(\gamma)_{|m|} (1)_{m_1} \ldots (1)_{m_N}} t^m. \tag{3.4}$$

We note that the series (3.4) is a generalization of the hypergeometric series Φ_1 given by Horns ([1]).

It is known that $S(1,\ldots,1;N)$ admits a family of solutions expressed by F_D. Let $\sigma^{(1)}_{m,n}$ $(m,n\in\mathbb{Z}_{\geqslant 0})$ be functions defined as follows:

$$\begin{aligned}\sigma^{(1)}_{0,n} &= 1,\\ \sigma^{(1)}_{1,n} &= (\theta_{N+2}-n)(\theta_{N+3}+n)t_1(1-t_1)^{-(\theta_{N+2}+\theta_{N+3}+1)}\\ &\quad\times F_D(-\theta_{N+3}-n,\theta_1,\ldots,\theta_N,-\theta_{N+1}-\theta_{N+3}-n+1;t).\end{aligned} \tag{3.5}$$

and

$$\sigma^{(1)}_{m,n} = \det\left(X^{i-1}Y^{j-1}\sigma^{(1)}_{1,n}\right)_{i,j=1,\ldots,m} \qquad (m\geqslant 2), \tag{3.6}$$

where

$$X = \frac{t_1}{t_1-1}\sum_{i=1}^{N}(t_i-1)D_i, \qquad Y = \frac{1}{t_1-1}\sum_{i=1}^{N}t_i(t_i-1)D_i. \tag{3.7}$$

Theorem 3.1 ([19]). *Let*

$$\tau_{(0,\ldots,0,m-n,m+n)} = C^{(1)}_{m,n}\sigma^{(1)}_{m,n} \qquad (m,n\in\mathbb{Z}_{\geqslant 0}), \tag{3.8}$$

where

$$C^{(1)}_{m,n} = t_1^{-m(m+1)/2}(1-t_1)^{m(\theta_{N+2}+\theta_{N+3}+m)}\prod_{k=1}^{m}\frac{1}{(\theta_{N+2}-n)_k}. \tag{3.9}$$

When $\rho=0$, this is a family of τ-functions for $S(1,\ldots,1;N)$.

Via (2.25) and taking a limit $\varepsilon\to 0$, each $\sigma^{(1)}_{m,n}$ is transformed into the function $\sigma^{(2)}_{m,n}$ defined as follows:

$$\begin{aligned}\sigma^{(2)}_{0,n} &= 1,\\ \sigma^{(2)}_{1,n} &= (\theta_{N+3}+n)t_1\,\mathrm{e}^{-t_1}\Phi_D(-\theta_{N+3}-n,\theta_2,\ldots,\theta_N,-\theta_{N+1}-\theta_{N+3}-n+1;t),\end{aligned} \tag{3.10}$$

and

$$\sigma^{(2)}_{m,n} = \det\left((t_1D_1)^{i-1}D^{j-1}_{N+3}\sigma^{(2)}_{1,n}\right)_{i,j=1,\ldots,m} \qquad (m\geqslant 2). \tag{3.11}$$

Thus we obtain the following theorem.

Theorem 3.2. *Let*

$$\tau_{(0,\ldots,0,m-n,m+n)} = C^{(2)}_{m,n}\sigma^{(2)}_{m,n} \qquad (m,n\in\mathbb{Z}_{\geqslant 0}), \tag{3.12}$$

where

$$C^{(2)}_{m,n} = t_1^{-m(m+1)/2}\,\mathrm{e}^{mt_1}. \tag{3.13}$$

When $\rho=0$, this is a family of τ-functions for $S(1,\ldots,1,2;N)$.

Recall the definition of the Schur polynomials. For each partition $\lambda=(\lambda_1,\ldots,\lambda_l)$, the Schur polynomial is a polynomial in $x=(x_1,x_2,\ldots)$ defined by

$$S_\lambda(x) = \det(p_{\lambda_i-i+j}(x))_{i,j=1,\ldots,l}, \tag{3.14}$$

where $p_n(x)$ are the polynomials defined as

$$p_n(x) = \sum_{k_1+2k_2+\cdots+nk_n=n}\frac{x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}}{k_1!k_2!\cdots k_n!}. \tag{3.15}$$

In a similar manner as [19], the τ-functions given by (3.12) are reduced to those expressed in terms of the Schur polynomials.

Theorem 3.3. *Let*

$$\tau_{(0,\dots,0,m-n,m+n)} = S_{(n^m)}(x) \quad (m, n \in \mathbb{Z}_{\geqslant 0}), \tag{3.16}$$

where we use the notation $(n^m) = (n, \dots, n)$ *and let*

$$x_1 = t_1 + \sum_{j=2}^{N+1} t_j\theta_j, \qquad x_k = \frac{1}{k}\sum_{j=2}^{N+1} t_j^k\theta_j \quad (k \geqslant 2). \tag{3.17}$$

When $\rho = \theta_{N+3} = 0$*, this is a family of* τ*-functions for* $S(1, \dots, 1, 2; N)$*.*

4. Algebraic solutions

In this section, we present a family of algebraic solutions expressed in terms of the universal characters.

We recall the definition of the universal character introduced by Koike [9], which is a generalization of the Schur polynomial. For each pair of partitions $[\lambda, \mu] = [(\lambda_1, \dots, \lambda_l), (\mu_1, \dots, \mu_{l'})]$, the universal character $S_{[\lambda,\mu]}(x, y)$ is defined as follows:

$$S_{[\lambda,\mu]}(x, y) = \det \begin{pmatrix} p_{\lambda_{l'-i+j}+i-j}(y), & 1 \leqslant i \leqslant l' \\ p_{\lambda_{-l'+i}-i+j}(x), & l'+1 \leqslant i \leqslant l+l' \end{pmatrix}_{1\leqslant i,j\leqslant l+l'}, \tag{4.1}$$

where $p_n(x)$ is the polynomial defined by (3.15).

The system $S(1, \dots, 1; N)$ admits a family of solutions expressed in terms of the universal characters. Let

$$\xi_i^2 = 1 - t_i \quad (i = 1, \dots, N). \tag{4.2}$$

Theorem 4.1 ([18, 20])**.** *Let*

$$\tau_{(0,\dots,0,m-n,0,m+n)} = N^{(1)}_{m,n} S_{[u!,v!]}(x, y) \quad (m, n \in \mathbb{Z}), \tag{4.3}$$

where

$$x_k = \frac{1}{k}\left(\theta_{N+2} + \sum_{i=1}^{N} \theta_i\xi_i^k\right), \qquad y_k = \frac{1}{k}\left(\theta_{N+2} + \sum_{i=1}^{N} \theta_i\xi_i^{-k}\right), \tag{4.4}$$

and

$$\begin{aligned} &[u!, v!] = [(u, u-1, \dots, 1), (v, v-1, \dots, 1)], \\ &u = |m+n-1/2| - 1/2, \qquad v = |m-n+1/2| - 1/2. \end{aligned} \tag{4.5}$$

When $\theta_{N+1} = 1/2$ *and* $\theta_{N+3} = -1/2$*, this is a family of* τ*-functions for* $S(1, \dots, 1; N)$*.*

Here we let

$$N^{(1)}_{m,n} = \prod_{i=1}^{N} \xi_i^{-\theta_i(\theta_i+2m-2n+1)/2} \prod_{i=1}^{N}\left(\frac{\xi_i+1}{2}\right)^{-\theta_i\theta_{N+2}} \prod_{i,j=1,i<j}^{N}\left(\frac{\xi_i+\xi_j}{2}\right)^{-\theta_i\theta_j}. \tag{4.6}$$

Via (2.25) and taking a limit $\varepsilon \to 0$, we obtain from theorem 4.1 the following theorem.

Theorem 4.2. *Let*

$$\tau_{(0,\dots,0,m-n,0,m+n)} = N^{(2)}_{m,n} S_{[u!,v!]}(x, y), \tag{4.7}$$

where

$$x_k = \frac{1}{k}\left(\theta_{N+2} - \frac{k}{2}t_1 + \sum_{i=2}^{N}\theta_i\xi_i^k\right), \qquad y_k = \frac{1}{k}\left(\theta_{N+2} + \frac{k}{2}t_1 + \sum_{i=2}^{N}\theta_i\xi_i^{-k}\right). \tag{4.8}$$

When $\theta_{N+1} = 1/2$ *and* $\theta_{N+3} = -1/2$*, this is a family of* τ*-functions for* $S(1,\ldots,1,2;N)$*.*

Here we let

$$N_{m,n}^{(2)} = \mathrm{e}^{\Delta_{m,n}}\prod_{i=2}^{N}\xi_i^{-\theta_i(\theta_i+2m-2n+1)/2}\prod_{i=2}^{N}\left(\frac{\xi_i+1}{2}\right)^{-\theta_i\theta_{N+2}}\prod_{i,j=2,i<j}^{N}\left(\frac{\xi_i+\xi_j}{2}\right)^{-\theta_i\theta_j}, \tag{4.9}$$

where

$$\Delta_{m,n} = \frac{t_1^2}{32} + \frac{t_1}{4}\left(2m-2n+1+\theta_{N+2}+\sum_{i=2}^{N}\frac{2\theta_i}{1+\xi_i}\right). \tag{4.10}$$

Remark 4.3. When $N = 1$, this is already given in [13].

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Appendix. Schlesinger transformations

In this appendix, we describe the action of the Schlesinger transformations for $S(1,\ldots,1,2;N)$ on the dependent variables.

The group of the Schlesinger transformations T_ν ($\nu \in L_2$) is generated by the transformations

$$\begin{aligned} T_1 &= T_{(0,\ldots,0,1,1)},\\ T_2 &= T_{(-1,0\ldots,0,1)},\\ T_3 &= T_{(0,-1,0,\ldots,0,1)},\\ &\vdots\\ T_{N+2} &= T_{(0,\ldots,0,-1,1)}. \end{aligned} \tag{A.1}$$

The action of T_k ($k = 1,\ldots,N+2$) on the dependent variables is described as follows:

$$\begin{aligned} T_1(A_1) &= R_2^{(1)}A_1E_2 + E_1A_1R_1^{(1)} - R_2^{(1)}A_{N+2}R_1^{(1)},\\ T_1(A_{N+2}) &= R_2^{(1)}A_{N+2}E_2 + E_1A_{N+2}R_1^{(1)} - E_1A_1E_2 + E_1R_1^{(1)} + \sum_{i=2}^{N+1}\frac{1}{t_i}R_2^{(1)}A_iR_1^{(1)},\\ T_1(A_j) &= R_2^{(1)}A_jE_2 + E_1A_jR_1^{(1)} - t_jE_1A_jE_2 - \frac{1}{t_j}R_2^{(1)}A_jR_1^{(1)} \quad (j = 2,\ldots,N+1), \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} R_1^{(1)} &= \frac{1}{(\theta_{N+3}+1)b_1}\begin{pmatrix} b_1\\ d_1\end{pmatrix}\begin{pmatrix}\theta_{N+3}+1 & b_\infty\end{pmatrix},\\ R_2^{(1)} &= \frac{1}{(\theta_{N+3}+1)b_1}\begin{pmatrix} -b_\infty\\ \theta_{N+3}+1\end{pmatrix}\begin{pmatrix}-d_1 & b_1\end{pmatrix}, \end{aligned} \tag{A.3}$$

for $k = 1$;

$$T_k(A_1) = E_1 A_1 R_1^{(k)} + R_2^{(k)} A_1 E_2 + t_k E_1 A_1 E_2 + \frac{1}{t_k} R_2^{(k)} A_1 R_1^{(k)},$$

$$T_k(A_{N+2}) = E_1 A_{N+2} R_1^{(k)} + R_2^{(k)} A_{N+2} E_2 + t_k E_1 A_{N+2} E_2 - E_1 A_1 E_2 + \frac{1}{t_k} R_2^{(k)} A_{N+2} R_1^{(k)} + \frac{1}{t_k^2} R_2^{(k)} A_1 R_1^{(k)},$$

$$T_k(A_k) = E_1 A_k R_1^{(k)} + R_2^{(k)} A_k E_2 - R_2^{(k)} E_2 - \frac{1}{t_k^2} R_2^{(k)} A_1 R_1^{(k)} - \sum_{i=2, i\neq k}^{N+2} \frac{1}{t_k - t_i} R_2^{(k)} A_i R_1^{(k)},$$

$$T_k(A_j) = E_1 A_j R_1^{(k)} + R_2^{(k)} A_j E_2 + (t_k - t_j) E_1 A_j E_2 + \frac{1}{t_k - t_j} R_2^{(k)} A_j R_1^{(k)} \quad (j \neq 1, k, N+2), \tag{A.4}$$

where

$$R_1^{(k)} = \frac{1}{(\theta_{N+3}+1)b_k} \begin{pmatrix} b_k \\ -a_k \end{pmatrix} \begin{pmatrix} \theta_{N+3}+1 & b_\infty \end{pmatrix}, \qquad R_2^{(k)} = \frac{1}{(\theta_{N+3}+1)b_k} \begin{pmatrix} -b_\infty \\ \theta_{N+3}+1 \end{pmatrix} (b_k \quad a_k), \tag{A.5}$$

for $k = 2, \ldots, N+1$;

$$T_1(A_1) = E_1 A_1 R_1^{(N+2)} + R_2^{(N+2)} A_1 E_2 - R_2^{(N+2)} A_{N+2} R_1^{(N+2)},$$

$$T_1(A_{N+2}) = E_1 A_{N+2} R_1^{(N+2)} + R_2^{(N+2)} A_{N+2} E_2 - E_1 A_1 E_2 - R_2^{(N+2)} E_2 + \sum_{i=2}^{N+1} \frac{1}{t_i} R_2^{(N+2)} A_i R_1^{(N+2)},$$

$$T_1(A_j) = E_1 A_j R_1^{(N+2)} + R_2^{(N+2)} A_j E_2 - t_j E_1 A_j E_2 - \frac{1}{t_j} R_2^{(N+2)} A_j R_1^{(N+2)} \quad (j = 2, \ldots, N+1), \tag{A.6}$$

where

$$R_1^{(N+2)} = \frac{1}{(\theta_{N+3}+1)b_1} \begin{pmatrix} b_1 \\ -a_1 \end{pmatrix} (\theta_{N+3}+1 \quad b_\infty), \qquad R_2^{(N+2)} = \frac{1}{(\theta_{N+3}+1)b_1} \begin{pmatrix} -b_\infty \\ \theta_{N+3}+1 \end{pmatrix} (b_1 \quad a_1), \tag{A.7}$$

for $k = N+2$.

References

[1] Erderyi A *et al* 1953 *Higher Transcendental Functions* (New York: MacGraw-Hill)

[2] Iwasaki K, Kimura H, Shimomura S and Yoshida M 1991 *From Gauss to Painlevé—A Modern Theory of Special Functions (Aspects of Mathematics* vol E16*)* (Braunschweig: Vieweg)

[3] Jimbo M, Miwa T and Ueno K 1981 Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I *Physica* D **2** 306–52

[4] Jimbo M and Miwa T 1981 Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II *Physica* D **2** 407–48

[5] Kawamuko H 1997 On the holonomic deformation of linear differential equations *Proc. Japan Acad. Ser. A Math. Sci.* **73** 152–4

[6] Kawamuko H 1997 On the polynomial Hamiltonian structure associated with the $L(1, g+2; g)$ type *Proc. Japan Acad. Ser. A Math. Sci.* **73** 155–7

[7] Kimura H 1989 The degeneration of the two dimensional Garnier system and the polynomial Hamiltonian structure *Ann. Mat. Pura Appl.* **155** 25–57

[8] Kimura H and Okamoto K 1986 On particular solutions of the Garnier systems and the hyper geometric functions of several variables *Q. J. Math.* **37** 61–80

[9] Koike K 1989 On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters *Adv. Math.* **74** 57–86

[10] Liu D 1997 Holonomic deformation of linear differential equations of A_g type and polynomial Hamiltonian structure *PhD Thesis* University of Tokyo

[11] Masuda T 2003 On a class of algebraic solutions to the Painlevé VI equation, its determinant formula and coalescence cascade *Funkcial. Ekvac.* **46** 121–71

[12] Masuda T 2004 Classical transcendental solutions of the Painlevé equations and their degeneration *Tohoku Math. J.* **56** 467–90

[13] Masuda T, Ohta Y and Kajiwara K 2002 A determinant formula for a class of rational solutions of Painlevé V equation *Nagoya Math. J.* **168** 1–25

[14] Okamoto K 1986 Isomonodromic deformation and Painlevé equations, and the Garnier system *J. Fac. Sci. Univ. Tokyo* **33** 575–618

[15] Okamoto K 1987 Studies on the Painlevé equations II *Japan J. Math.* **13** 47–76

[16] Okamoto K 1999 The Hamiltonians associated with the Painlevé equations *The Painlevé property: One Century Later (CRM Series in Mathematical Physics)* ed R Conte (Berlin: Springer)

[17] Suzuki T 2005 Affine Weyl group symmetry of the Garnier system *Funkcial. Ekvac.* **48** 203–30

[18] Tsuda T 2003 Universal characters and Integrable systems *PhD Thesis* (University of Tokyo)

[19] Tsuda T 2003 Rational solutions of the Garnier system in terms of Schur polynomials *Int. Math. Res. Not.* **43** 2341–58

[20] Tsuda T 2004 Toda equations and special polynomials associated with the Garnier system *Adv. Math.* at press (*Preprint* nlin.SI/0408060)